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LETTER TO THE EDITOR

Gauge geometry of financial markets

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Abstract. In this paper we show that financial markets have their own intrinsic underlying geometrical structure: fibre bundle geometry. This structure allows one to formulate a local gauge symmetry of rescaling of asset units in geometrical terms. We take this into account during the course of model construction, thus providing financial economics with physical methodology.

1. Introduction

The development of mathematical tools of economics is entangled with developments of economics itself. Demand–supply curves, evolutionary differential equations, models of general equilibrium, game theory and stochastic calculus are just a few examples of advances in the mathematical machinery of economics over the years. This paper aims to add a new 'building block' to the mathematical construction of contemporary economics: namely, we show how abstract objects of differential geometry arise naturally in the context of exchange economy.

It is not the first time that notions of (stochastic) differential geometry have appeared in finance. Modelling of the dynamics of market prices as a random work on multi-dimensional state space prompts the analogy (see Hughston 1994). As soon as a state space is considered as a smooth manifold the use of Riemann metrics and connections is natural. This allows us to rewrite known pricing equations in terms of geometrical covariant quantities. What we suggest in this paper is, however, quite different. We argue that the general exchange economy, and financial markets in particular, where wealth is transformed from one form to another according to exchange rules, are natural realizations of formal definitions of a fibre bundle space equipped with a connection, or rule of 'parallel transport'.

The paper is organized as follows. In the next section we informally introduce the fibre bundle spaces and show using simple examples how they appear in a financial setting. In section 3, the formal definitions of fibre bundles and related relevant objects are given. These definitions are used later in section 4 to describe a formal construction of the fibre bundle space for financial markets. Using the results of this section it is possible to formulate a market dynamics as an evolution of a dynamical system in the fibre bundle. Section 5 is devoted to the notion of a special symmetry, namely gauge symmetry, and discusses a possibility for the dynamics to be gauge invariant. The symmetry allows us to reduce a number of legitimate

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market models and considerably simplifies their study. The last section contains concluding remarks.

2. Simple examples

We start with very basic examples which will, however, elucidate the formal constructions of the following sections. At the moment, formal definitions are not necessary. The purpose of this section is to give a vague idea as to what the fibre bundle space is. It is possible to imagine the fibre bundle space as a set of spaces (fibres) which are parametrized by points of a set called the base. In brief, a naive picture of the fibre bundle gives the fibre bundle E as a union $\cup_{x\in B} F_x$ of fibres F_x which correspond to points x of the base B. In what follows, one can consider the fibre bundle E as a direct product of $B \times F$. This means that the points of the fibre bundle can be presented as (x, f), where the first number is the coordinate in the base and the second one is the coordinate in the corresponding fibre.

Imagine that we are watching a particle moving in such a fibre bundle. The particle can move inside the fibre as well as between fibres. Putting aside the issue of coordinates on the base we can pose the question: 'What if coordinates in different fibres are not adjusted to each other?' In this case, the changing of coordinates does not say anything about the real change of particle position which is characterized by the change of 'real' coordinates—exactly like the rate of return, this does not say anything until inflation is accounted for and the real rate of return is calculated by Fisher's formula. As in the example with inflation, we must subtract from the total change of the coordinate its superficial change which is associated with zero real change and which is determined by coordinate disagreement in different fibres. It is this superficial change that determines the rule for coordinate comparison in coordinate systems of different fibres or, as mathematicians put it, parallel transport.

Now let us turn to finance for examples. We assume that there are two currencies (two points on the base) and we wish to compare \$4 and £3 (numbers 3 and 4 in coordinate systems of fibres corresponding to points 'dollar' and 'pound' on the base). At a first glance, four notes seem more attractive than three. However, everyone would prefer to have £3 because at an exchange rate of \$1.67 per £1 can get \$5 for these £3, which is much better than to have only four. We can see that when assets are transformed from \$4 into £3 the real change was equal to +1 dollar instead of initial -1. In our case, the real value of \$4 was £2.40, and the fictitious change was equal to £1.60. When compared, £2.40 differs from £3 by 60p, which is equivalent to exactly \$1. For a mathematician, all this would mean is that £2.40 is equal to \$4 under parallel transport from one point on the base to another, and a covariant (real) difference of £3 and \$4 is equal to 60p.

Net present value gives us another financial example of parallel transport. Assume that we can choose between £100 now, or £103 a year later. At first glance 103 (the same currency!) notes seem more attractive than 100. However, a reader is likely to choose 100 because at an interest rate of 5%, £100 will become £105 in a year, which is definitely better than 103. Instead of counting all pounds a year later we can count them now. Then, we will be able to compare £100 with the discounted value of £103: that is, the net present value equal to $98.10 = 103/(1 + 0.05)$. Again, one can see that as the assets move in time from pounds at present to pounds a year later, the real change is £2 instead of the initial +3, the parallel translation of £100 amounting to £105 and the covariant (real) difference being 2.

After we have defined parallel transport, we can address the issue of the difference in the results of parallel transport carried out by different routes on the base which have a common beginning and end. The financial meaning of the difference is quite transparent: this is the excess rate of return on the arbitrage operation (with the condition of prior knowledge of prices) which consists of opening a short and a long position at the initial point and closing the positions at the end point of the path. The difference between the initial value of the transported amount and its value after parallel transport along the closed path mathematically determines the curvature tensor of the fibre bundles. Therefore, the notion of the curvature tensor is a natural and convenient quantity to present a mispricing in this new language.

To sum up, we can say that when financiers buy and sell securities, exchange currency or calculate net present values, they make parallel transports in fibre bundles. What is more, they have been doing this for hundreds of years without being aware of it.

3. Fibre bundles: formal definitions

This section aims to introduce fibre bundles to the reader since these structures are not yet common in economic literature. We give formal definitions of fibre bundles and related objects which are relevant for our purposes. That is why, despite the fact that the section looks a bit 'dry' because of the amount of definitions, it will be helpful at least to look through it, if not to read it carefully.

The fibre bundle is a very popular mathematical structure and there exists a vast literature on the subject. Further details can be found, for example, in Husemoller (1975), Dubrovin *et al* (1984) and references therein.

Definition 1 (Dubrovin *et al* 1984). *A smooth fibre bundle is a composite object, made up of:*

(1) a smooth manifold E called the total (bundle) space;
(2) a smooth manifold B called the base space;
(3) a smooth surjective map $p : E \to B$ called the projection whose Jacobian is required to have maximal rank $n = \dim B$ at every point;
(4) a smooth manifold F called the fibre;
(5) a Lie group G of smooth transformations (self-diffeomorphisms) of the fibre F (it implies that the action $G \times F \to F$ is smooth on $G \times F$): this group is called the structure group of the fibre bundle;
(6) a 'fibre bundle structure' linking the above entities, defined as follows. The base B comes with a particular system of local coordinate neighbourhoods U_α (called the coordinate neighbourhoods or charts), above each of which the coordinates of the direct product are introduced via a diffeomorphism $\phi_\alpha : F \times U_\alpha \to p^{-1}(U_\alpha)$ satisfying $p\phi(y, x) = x$; the transformations $\lambda_{\alpha\beta} = \phi_\beta^{-1}\phi_\alpha : F \times U_{\alpha\beta} \to F \times U_{\alpha\beta}$, where $U_{\alpha\beta} = U_\alpha \cap U_\beta$ are called the transition functions of the fibre bundle. Every transformation $\lambda_{\alpha\beta}$ has the form

$$\lambda_{\alpha\beta}(y, x) = (T^{\alpha\beta} y, x)$$

where for all α, β, x the transformation $T^{\alpha\beta}(x)$ is an element of the structural group G.

To complete the definition with an intuitive picture one can imagine a fibre bundle as a union of fibres $F_b = p^{-1}(b)$, for any element $b \in B$. This union is parametrized by the base B and 'glued together' by the topology of the space E using the transition functions $\lambda_{\alpha\beta}$. The total space E can be represented as $E = \cup_\alpha F \times U_\alpha$. For each of the charts U_α the coordinate system in the fibre F can be chosen independently. This generates two independent coordinate systems in F for each overlapped neighbourhood: $U_{\alpha\beta} = U_\alpha \cap U_\beta$. To identify equivalent points (to 'glue them together') in different coordinate systems one uses elements of the structural group G stating that

$$T^{\alpha\beta} y|_{U_\alpha} = y|_{U_\beta} \qquad y \in F.$$

From the definition of functions $T^{\alpha\beta}$ it follows that:

$$T^{\alpha\beta}(x) = (T^{\beta\alpha}(x))^{-1} \qquad \text{and} \qquad T^{\alpha\beta}(x)T^{\beta\gamma}(x)T^{\gamma\alpha}(x) = 1$$

where the second equation is understood as holding on the region of intersection $U_\alpha \cap U_\beta \cap U_\gamma$.

Definition 2. *A fibre bundle ξ is trivial with the fibre F if ξ is isomorphic with the fibre bundle $(B \times F, p, B)$. In what follows, all financial examples we study will be globally trivial.*

Definition 3. *A principal fibre bundle is defined to be a fibre bundle whose fibre F coincides with the structural group, which acts on the fibre $F = G$ by right translations $R_g : G \to G$, $R_g(x) = xg$.*

Now we introduce a connection which allows one to use a differential calculus on a fibre bundle. A fibre bundle with connection can be imagined as a family $\{F_b\}$ of fibres (whose union $\cup_b F_b$ is the total space E) which is also provided with a rule of 'parallel transport'. Given any path $\gamma(t)$, $a \leqslant t \leqslant b$, in the base B the connection defines a rule for 'parallel transporting' the fibre F along the path $\gamma(t)$ from one end to the other, i.e. a map

$$\phi_\gamma : F_{\gamma(a)} \to F_{\gamma(b)}$$

satisfying the following natural requirements:

(1) $\phi(\gamma)$ depends continuously on the path $\gamma(t)$;
(2) $\phi(\gamma)$ is independent on the parametrization of the path;
(3) $\phi(\gamma)$ is the identity map if $\gamma(t) = \text{const}$;
(4) the following equations take place:

$$\phi(\gamma_1\gamma_2) = \phi(\gamma_1)\phi(\gamma_2) \qquad \phi(\gamma^{-1}) = (\phi(\gamma))^{-1}.$$

The connection is the G-connection if the map ϕ_γ is defined as

$$\phi_\gamma F_{\gamma(a)} = g(\gamma)F_{\gamma(b)} \qquad g(\gamma) \in G \forall\gamma.$$

The properties of the map ϕ_γ imply the same properties for the function $g(\gamma)$.

Definition 4. *The gauge transformation with a function $g(x) \in G$ is defined as follows:*

$$F_x \to g(x)F_x \qquad \forall x \in B.$$

The gauge transformation can be thought of as a point-dependent change of the coordinate system in the fibre for each point of the base. Under gauge transformation, the G-connection $g(\gamma)$ is transformed in a very simple way:

$$g(\gamma) \to g(\gamma(b))g(\gamma)g^{-1}(\gamma(a))$$

for any path $\gamma(t)$, $a \leqslant t \leqslant b$.

Using $g(\gamma)$ it is possible to introduce the parallel transport of an element of the fibre along the path γ: one calls the expression $g(\gamma)\psi \in F_{\gamma(b)}$ the result of the parallel transport of $\psi \in F_{\gamma(a)}$.

Definition 5. *The cross section of the fibre bundle is a map $\psi : B \to E$, such that $p\psi = 1_B$: i.e., $\psi(x) \in F_x$ for each x.*

The next step is to introduce the covariant difference of values of a cross section $\psi(x)$ as

$$\Delta_\gamma\psi = \psi(\gamma(b)) - g(\gamma)\psi(\gamma(a)).$$

If γ is an infinitesimal path connecting points x and $x + \mathrm{d}x$ in the base then the covariant difference transforms to the covariant derivative $\mathrm{D}\psi$:

$$\mathrm{D}\psi(x) \equiv \sum_{\mu=1}^{\dim B} \mathrm{D}_\mu \psi(x) \mathrm{d}x^\mu = \sum_{\mu=1}^{\dim B} \left(\frac{\partial \psi(x)}{\partial x^\mu} - A_\mu(x)\psi(x) \right) \mathrm{d}x^\mu$$

where $g(\gamma) = 1 + \sum_{\mu=1}^{\dim B} A_\mu(x)\mathrm{d}x^\mu$ and $A_\mu(x)$ are elements of the Lie algebra of the structural group G.

As is clear from the definition, the result of the parallel transport depends not only on end points $\gamma(a)$ and $\gamma(b)$ but on the whole path γ. This means that the results of the parallel transports with the same end points but different paths, can be different. The *curvature* is a measure of this difference. To define it one considers two paths γ_1 and γ_2 with the same end points and combines them in a composite cyclic path $\gamma_2^{-1}\gamma_1$ which consists of path γ_1 going from $\gamma(a)$ to $\gamma(b)$ and the return path γ_2^{-1} from point $\gamma(b)$ to the original point $\gamma(a)$. If the paths are infinitesimal then $g(\gamma_2^{-1}\gamma_1)$ can be expanded as

$$g(\gamma_2^{-1}\gamma_1) = 1 - \sum_{\mu,\nu=1}^{\dim B} \tfrac{1}{2} F_{\mu,\nu}\sigma^{\mu\nu}$$

where σ is a bivector corresponding to the surface encircled by the path $\gamma_2^{-1}\gamma_1$ and F is the *curvature tensor*.

4. Financial market as a fibre bundle

In this section we show that a financial market represents a structure of the abstractions introduced in the previous section. More precisely, we give a description of the relevant fibre bundles, construct the parallel transport rules using for this elements of the structural group, and give an interpretation of the parallel transport operators. The corresponding curvature is also defined and it is shown to be equal to the rate of excess return on the elementary plaquette arbitrage operation. This opens the way for a construction of the dynamics of parallel transport factors which provides the lattice gauge theory formulation.

Theorem 1. *If X is a finite set of assets evolving in time such that there exists at least one asset in X which can be exchanged with any other asset in X at any time, then X possesses a structure of a trivial fibre bundle with a connected base. The connectivity is defined by the rule of the asset's exchange.*

Proof. To prove the theorem we construct the fibre bundle for an arbitrary set X.

First, we construct a base of the fibre bundle which is the main nontrivial step. Let us order the complete set of assets X and label them by numbers from 0 to N. This set can be represented by N (asset) points on a two-dimensional plane (the dimension is a matter of convenience and can be chosen arbitrarily). To introduce time into the construction we attach a copy of the Z-lattice (i.e. a set of all integer numbers $\{\ldots, -1, 0, 1, 2, \ldots\}$) to each asset point. We use discretized time since there is a natural time step and all real trades happen discretely. Taken together this gives the base set $L = \{1, 2, ..., N\} \times Z$.

The next step in the construction is to define the *connectivity* of the base which is the key element to define a curve in L. To do this, we start with an introduction of a matrix of links $\Gamma : L \times L \to \{0, \pm 1\}$ which is defined by the following rule: for any $x \equiv (i, n) \in L$ and $y \equiv (k, m) \in L$: $\Gamma(x, y) = 0$, except for

(1) $i = k$ and $n = m - 1$, accepting that the ith security exists at the nth moment and this moment is not an expiry (termination) date for the security;
(2) $n = m$ and at the nth moment of time the ith asset can be exchanged on some quantity of the kth asset and at some rate.

In latter situations: $\Gamma(x, y) = 1 = -\Gamma(y, x)$.

Using the matrix $\Gamma(.,.)$ we define a *curve* $\gamma(x, y)$ in L which links two points $x, y \in L$. We call the set $\gamma(x, y) \equiv \{x_j\}_{j=1}^{p}$ a curve in L_0 with ends at points $x, y \in L$ and $p - 1$ segments if $x = x_1, x_p = y, \forall x_j \in L$ and

$$\Gamma(x_j, x_{j+1}) = \pm 1 \qquad \text{for} \quad \forall j = 1, \ldots, p - 1.$$

This base L is connected since there is at least one asset which can be exchanged with any other asset and thus can serve as a link in the curve connecting any two assets. This completes the construction of the base of the fibre bundle.

The next step is to define the corresponding structure group. The structural group G to be used is a group of dilatations: i.e., the group of multiplications by positive real numbers. The corresponding irreducible representation is the following: the group G is a group of maps g of $R_+ \equiv]0, +\infty)$ to R_+, which act as a multiplication of any $x \in R_+$ on some positive constant $\lambda(g) \in R_+$:

$$g(x) = \lambda(g) \cdot x.$$

Transition functions of a fibre bundle with the structure group correspond below to various swap rates, exchange rates and discount factors for assets.

The last step in the construction is the definition of fibres. In this paper we use fibre bundles with the following fibres F:

(1) $F = G$: i.e., the fibre coincides with the structure group. The corresponding fibre bundle is called the principal fibre bundle E_P. The dynamical theory on the fibre bundle corresponds to a dynamics of prices and rates.
(2) $F = R_+$: this fibre bundle will be important to describe cash–debt flows. Indeed, a cross section s (a rule which assigns a preferred point $s(x)$ on each fibre to each point $x = (i, m) \in L$ of the base) of the fibre bundle gives the number of units of the ith asset at the moment of time m.

The fibre bundle E is defined now as a trivial one, i.e. $E = L \times F$. We do not concern ourselves with the definition of the projections. This completes the construction and the proof. □

Any theory of the financial market can now be formulated in terms of these cross sections of fibre bundles. A theory can, therefore, be constructed in such a way that all geometrical properties of the objects (such as covariance under gauge transformations, which corresponds to a change of money units) will be honoured.

It is straightforward to define connectivity, links matrices and bases for fibre bundles for the simple stock exchange with N tradable securities, the foreign-exchange market and financial derivatives.

Theorem 2. *Prices and rates of return define a natural connection in the fibre bundle E.*

Proof. As was defined in the previous section, the G-connection is a rule of the parallel transport of an element of a fibre from one point (x) of a base to another point (y). This means that an operator of the parallel transport along the curve γ, $U(\gamma) : F_x \to F_y$ is an element of the structural group of the fibre bundle. Since we do not deal with the continuous case and restrict ourselves to lattice formulation, we do not need to introduce a vector field of the connection but rather must use elements of the structural group G itself. By definition, an operator of the parallel transport along a curve γ, $U(\gamma)$, defined as a product of operators of the parallel transport along the links which constitute the curve γ:

$$U(\gamma) = \prod_{i=1}^{p-1} U(x_i, x_{i+1}) \qquad \gamma \equiv \{x_i\}_{i=1}^{p} \qquad x_1 = x \qquad x_p = y.$$

This means that we need to define only the parallel transport operators along elementary links. Since $U(\gamma) = U^{-1}(\gamma^{-1})$, this restricts us to a definition only of those along elementary links with positive connectivity. Summing up, the rules of parallel transport in the fibre bundles are completely defined by a set of parallel transport operators along elementary links with positive connectivity. The definition of the set is equivalent to a definition of parallel transport in the fibre bundle.

In the proof of theorem 1, connectivity was defined as the possibility of asset movements in 'space' and time: it allows us to give the following interpretation of the parallel transport. We have defined two principal kinds of links with positive connectivity. The first one connects two points (i, n) and $(i, n + 1)$ and represents a deposition of the ith asset for one unit of time. This deposition then results in a multiplication of the number of asset units by an interest factor (or internal rate of return factor) calculated as

$$U((i, n), (i, n + 1)) = \mathrm{e}^{r_i \Delta} \in G$$

where Δ is a time unit and r_i is an appropriate rate of return for the ith asset. In the continuous limit r_i becomes a time component of the corresponding connection vector field at the point $(i, \Delta n)$.

In the same way, the parallel transport operator is defined for the second kind of elementary links, i.e. links between (i, n) and $(k, n+1)$ if there is a possibility to change at the nth moment a unit of the ith asset on $S_n^{i,k}$ units of the kth asset:

$$U((i, n), (k, n + 1)) = S_n^{i,k} \in G.$$

Here $S_n^{i,k}$ is the price of the ith asset in terms of the kth asset. In general, an operator of the parallel transport along a curve is a multiplier by which a number of asset units is multiplied as a result of an operation represented by the curve. □

Results of parallel transports along two different curves with the same boundary points are not equal for a generic set of the parallel transport operators. A measure of the difference is the curvature tensor F. Its elements are equal to the resulting change in the multiplier due to a parallel transport along a loop around an infinitesimal elementary plaquette with all nonzero links in the base L:

$$F_{\text{plaquette}\to 0} = \prod_m U_m - 1.$$

The index m runs over all plaquette links, $\{U_m\}$ are corresponding parallel transport operators and an agreement about an orientation is implied.

Theorem 3. *The elements of the curvature tensor are excess returns on the operation corresponding to a plaquette.*

Proof. Since elements of the curvature tensor are local quantities, it is sufficient to consider an elementary plaquette on a 'space'–time base graph. Let us, for example, consider two different assets (for the moment we will call them share and cash) which can be exchanged with each other by some exchange rate S_i (one share is exchanged to S_i units of cash) at some moment

T_i, and the reverse rate (cash to share) is S_i^{-1}. We suppose that there exists a characteristic time step Δ and this is taken as a time unit. So the exchange rates S_i are quoted on a set of the equidistant times: $\{T_i\}_{i=1}^N$, $T_{i+1} - T_i = \Delta$. The interest rate for cash is r_1, so that between two subsequent times T_i and T_{i+1} the volume of cash is increased by a factor $e^{r_1\Delta}$. The shares are characterized by a rate r_2.

Let us consider an elementary (arbitrage) operation between two subsequent times T_i and T_{i+1}. There are two possibilities for an investor, who possesses a cash unit at moment T_i, to obtain shares by moment T_{i+1}. The first one is to deposit cash into a bank with interest rate r_1 at moment T_i, withdraw money back at moment T_{i+1} and buy shares for price S_{i+1} each. In this way the investor gets $e^{r_1\Delta}S_{i+1}^{-1}$ shares at moment T_{i+1} for each unit of cash he had at moment T_i. The second way is to buy the shares for price S_i each at moment T_i. Then, at moment T_{i+1}, the investor will have $S_i^{-1}e^{r_2\Delta}$ shares for each unit of cash at moment T_i. If these two numbers ($e^{r_1\Delta}S_{i+1}^{-1}$ and $S_i^{-1}e^{r_2\Delta}$) are not equal then there is a possibility for arbitrage. Indeed, suppose that $e^{r_1\Delta}S_{i+1}^{-1} < S_i^{-1}e^{r_2\Delta}$, then at moment T_i an arbitrager can borrow one unit of cash, buy S_i^{-1} shares and get $S_i^{-1}e^{r_2\Delta}S_{i+1}$ units of cash from selling shares at moment T_{i+1}. The value of this cash discounted to moment T_i is $S_i^{-1}e^{r_2\Delta}S_{i+1}e^{-r_1\Delta} > 1$. This means that $S_i^{-1}e^{r_2\Delta}S_{i+1}e^{-r_1\Delta} - 1$ is an arbitrage excess return on the operation. On the other hand, as we have shown above, this represents lattice regularization of an element of the curvature tensor along the plaquette. Similarly, one can consider the case $e^{r_1\Delta}S_{i+1}^{-1} > S_i^{-1}e^{r_2\Delta}$ and 'space'–'space' plaquettes. □

Let us consider the following quantity:

$$(S_i^{-1}e^{r_2\Delta}S_{i+1}e^{-r_1\Delta} + S_i e^{r_1\Delta}S_{i+1}^{-1}e^{-r_2\Delta} - 2)/2\Delta. \tag{1}$$

This is the sum of excess returns on the plaquette arbitrage operations. In the continuous limit this quantity converges, as usual, to a square of the curvature tensor element. The absence of arbitrage is equivalent to the equality

$$S_i^{-1}e^{r_2\Delta}S_{i+1}e^{-r_1\Delta} = S_i e^{r_1\Delta}S_{i+1}^{-1}e^{-r_2\Delta} = 1$$

and we can use quantity (1) to measure the arbitrage (excess rate of return). In a more formal way, expression (1) may be written as

$$R = (U_1U_2U_3^{-1}U_4^{-1} + U_3U_4U_2^{-1}U_1^{-1} - 2)/2\Delta.$$

In this form it can be generalized for other plaquettes such as, for example, the 'space'–'space' plaquettes.

The last point to add in this section is the notion of gauge transformation. Gauge transformation means a local change of a scale in the fibres:

$$f_x \to g(x)f_x \equiv f'_x \qquad f_x \in F_x \qquad g(x) \in G \qquad x \in E$$

together with the following transformation of the parallel transport operators:

$$U(y,x) \to g(y)U(y,x)g^{-1}(x) \equiv U'(y,x) \in G.$$

It is easy to see that the parallel transport operation commutes with a gauge transformation

$$g(y)(U(y,x)f_x) = U'(y,x)f'_x \tag{2}$$

and that the curvature tensor is invariant under the transformation

$$U_1U_2U_3^{-1}U_4^{-1} = U'_1U'_2(U'_3)^{-1}(U'_4)^{-1}. \tag{3}$$

As we will see in the next section, this is a very important property for the modelling of financial market dynamics.

5. Gauge invariance

In this section we introduce a very important symmetry: namely, the gauge symmetry or symmetry with respect to arbitrary gauge transformations, which will play a significant role in model building.

Let us ask ourselves the following question: 'If tomorrow we measure money not in pounds but in pence and adjust prices accordingly, will the market dynamics be different?'. Instead of pence one could use 10p but does it matter? If one starts to count shares in hundreds of shares rather than in shares will it change the trading? If the answer to these questions is 'no' then the financial market obeys the symmetry with respect to a local change of asset units or, in geometrical terms, with respect to gauge transformations.

Proposition 1. *The symmetry of the financial dynamics with respect to arbitrary changes of numbers for any asset at any moment of time constitutes the gauge symmetry in the fibre bundle E constructed in theorem 1 with the connection defined in theorem 2.*

There is no doubt that the real world has this property, at least to a certain extent: agents do not start behaving in a different way only because they are dealing with 100p instead of pounds or if there are two lots of 50 shares instead of one lot of 100 shares. This means that there are no fundamental asset units. At first sight, the symmetry is not that powerful and is almost trivial. However, this first impression is wrong. To start with, the symmetry group is actually local and, hence, infinite-dimensional since the dilatations of the asset units are allowed for any asset and, importantly, for *any moment of time*. Second, in building a theory to describe a financial market dynamics one can only use mathematical objects that remain unaltered when the units of measurement are changed: i.e., gauge invariant objects such as curvature tensor and blocks of the covariant derivatives or covariant differences.

There is no perfect symmetry in nature. Gauge symmetry is not perfect either. Transaction costs and bid–ask spread do violate the symmetry (see Loeb 1983). Furthermore, as has been mentioned by many authors (see Fama *et al* 1969, Grinblatt 1984, Bar-Yosef *et al* 1977, Charest 1978, Conroy *et al* 1990, Copeland 1979), share splits are not perfect symmetry operations in real life and generally do change the effective price. These imperfections exist and could form the subject of a separate study. However, it is reasonable to start with the ideal problem and introduce the market imperfections as a perturbation later.

6. Conclusion

In this paper we suggested a geometrical framework to describe financial markets. It is clear that the picture can be generalized for any exchange economy system. Indeed, the only important issues for the above constructions were the existence of exchangable assets and rules of this exchange. As soon as the picture stays the same for any such system, the general rules of model construction stay the same also. This may open up a possibility to overcome equilibrium market model restrictions and help to develop, from first principles, dynamical economic theories.

The gauge symmetry of financial markets itself is a very interesting property (see Young (1999), Bak *et al* (1999) for other discussions of financial symmetries with respect to changes in numbers). But what is really exciting about the underlying geometrical picture is the possibility to develop dynamical models of prices and money flows which are similar to well known physical models. The reader can find more on the applications of gauge symmetry to real markets in Ilinski (1997, 1998), Ilinski and Kalinin (1997), Ilinski and Stepanento (1998), Ilinskaia and Ilinski (1998).

In conclusion, we wish to add that this is not first time that abstract definitions of fibre bundle geometry have appeared in the description of real life. Many important concepts in physics can be interpreted in terms of the geometry of fibre bundles (Eguchi 1980). Maxwell's theory of electromagnetism and Yang–Mills' theories are essentially theories of the connections on principal bundles with a given gauge group G as the fibre. Einstein's theory of gravitation deals with the Levi-Civita connection on the frame bundle of the space–time manifold. Actually, the original Hermann Weyl gauge theory (Weyl 1919, Moriyasu 1983), the oldest gauge theory, which he suggested to explain electromagnetism, was very close to the theory we have begun to develop here. One hopes that by using gauge theory, economics could eventually become a much more exact science, as happened for electrodynamics.

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